Mathematical jewels that you may not have seen in school

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The discovery of irrational numbers

The Platonic Solids

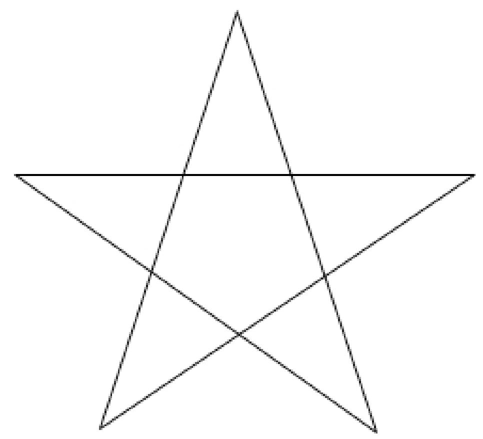
The Archangel Gabriel's horn

Intersecting a cone

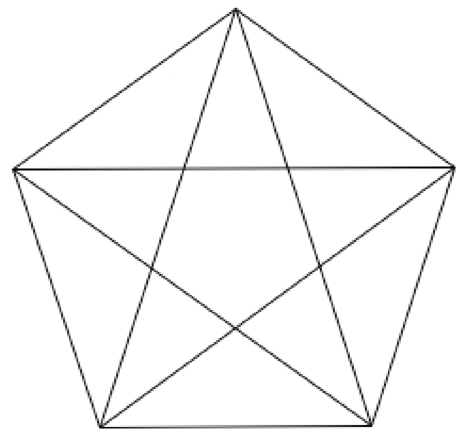
Three amazing sequences

1. THE PYTHAGOREAN HUGIEIA

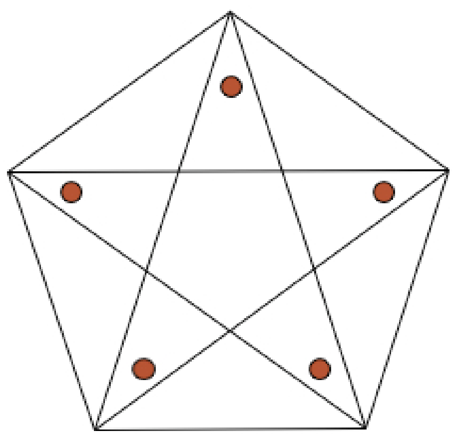
Was the symbol of health
and perfection

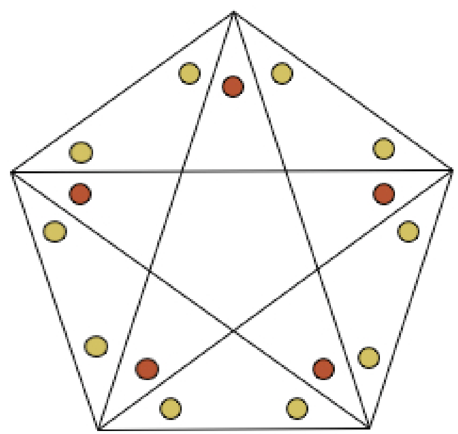


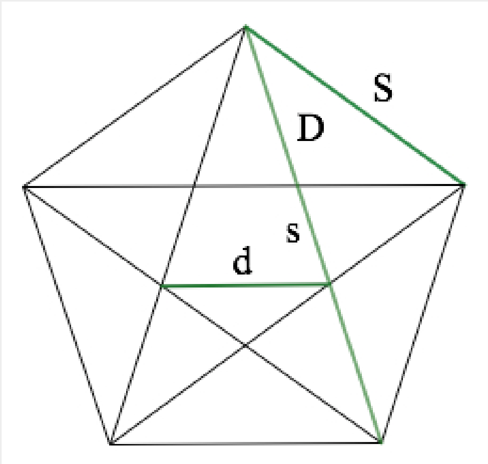
The hugieia generated the pentagon



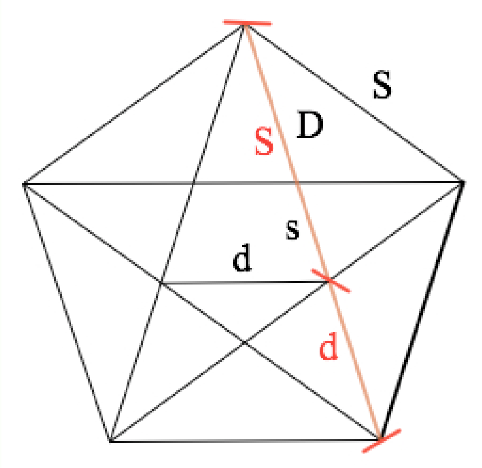
In the pentagon, Híppasos of Metapontum (5th century B.C.) discovered the concept of irrationality



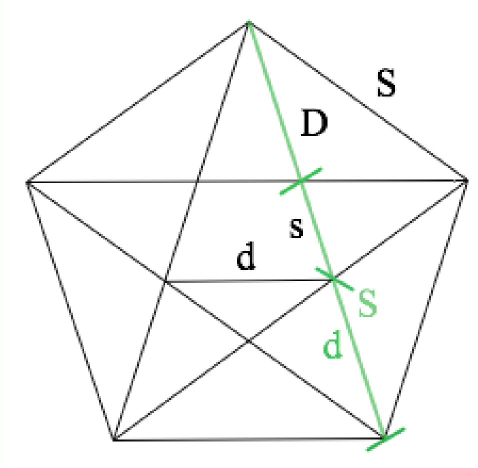




Consider side S
and diagonal D



$D = S + d \quad (1)$



$S = d + s \quad (2)$

THEN D AND S ARE INCOMMENSURABLE

(i.e. the ratio of their lengths is an irrational number)

Do you see why?

Reductio ad absurdum

Recall: $D = S + d$ (1) $S = d + s$ (2)

Let k be an arbitrary unit of measure (a segment).

Let $|D| = |k|\,x$, $|S| = |k|\,y$ (x, y integers).

Then $|d| = |k|\,w$ by (1), $|s| = |k|\,z$ by (2):

but this is impossible because the partition of the pentagon can go on for ever: so, for any given k, we reach a pentagon side s' with $|s'| < |k|$.

Incidentally $|D| / |S|$ is the golden ratio

2. THE PLATONIC SOLIDS

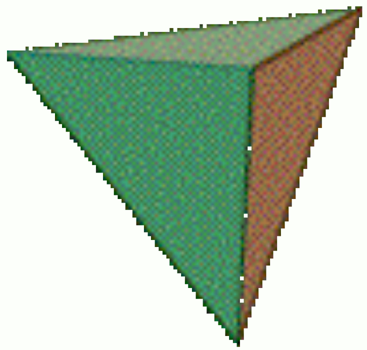
In the year 360 B.C. Plato wrote one of his last dialogues, entitled Timaeus, on the nature of the physical world.

(in the translation by Benjamin Jowett)

"I have now to speak of their (i.e., of the physical entities) several kinds, and show out of what combinations of numbers each of them was formed".

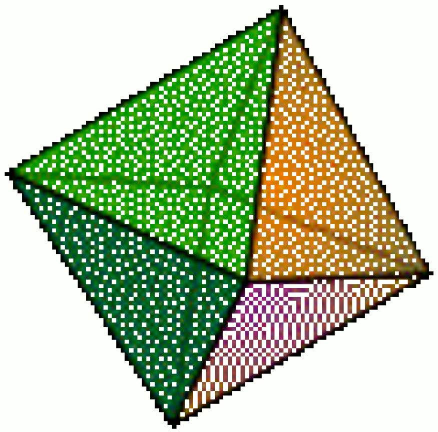
“Four equilateral triangles, if put together, make out of every three plane angles one solid angle, being that which is nearest to the most obtuse of plane angles; and out of the combination of these four angles (i.e., the vertices) arises the first solid form which distributes into equal and similar parts the whole circle in which it is inscribed”.

TETRAHEDRON { 3,3 }



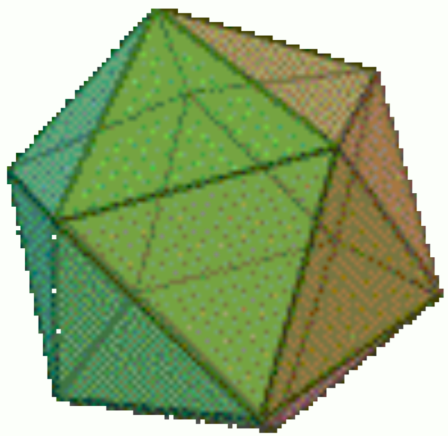
"The second species of solid is formed out of the same triangles, which unite as eight equilateral triangles and form one solid angle out of four plane angles, and out of six such angles the second body is completed".

OCTAHEDRON { 3,4 }



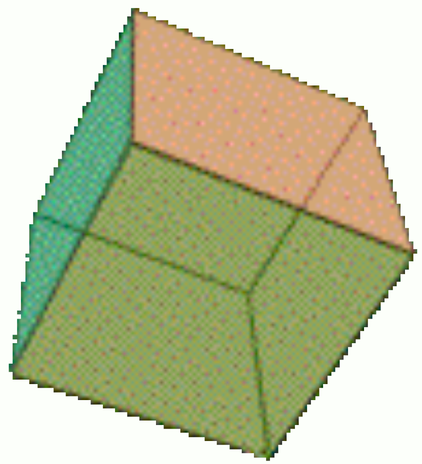
“And the third body is made up of triangular elements, forming twelve solid angles, each of them included in five plane equilateral triangles, having altogether twenty bases, each of which is an equilateral triangle”.

ICOSAHEDRON { 3,5 }



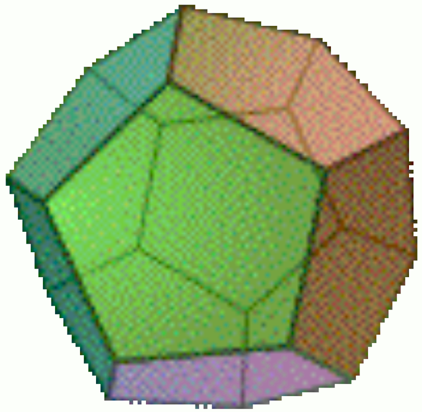
"Six of these (quadrangles) united form eight solid angles, each of which is made by the combination of three plane right angles; the figure of the body thus composed is a cube, having six plane quadrangular equilateral bases".

CUBE { 4,3 }



"There was yet a fifth combination which God used in the delineation of the universe".

DODECAHEDRON { 5,3 }



"To earth, then, let us assign the cubical form; for earth is the most immoveable of the four and the most plastic of all bodies, and that which has the most stable bases must of necessity be of such a nature".

"Of the remaining forms . . . . we assign the acutest body (the tetrahedron) to fire, and the next in acuteness (the octahedron) to air, and the third (the icosahedron) to water. Both according to reason and probability".

Plato's disciples later associated the dodecahedron to ether.

"We must imagine all these to be so small that no single particle of any of the four kinds is seen by us on account of their smallness: but when many of them are collected together their aggregates are seen".

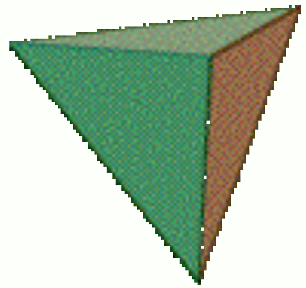
Descartes in the 1600's, Euler in the 1700's, and Cauchy in the 1800's studied polyhedra again. We consider "Euler's formula" and its prove given by Cauchy.

$$V - E + F = k$$

where k is the Euler's characteristic

for polyhedra without holes $k = 2$

e.g. in $\{ 3,3 \}$ we have $V = 4$, $E = 6$, $F = 4$

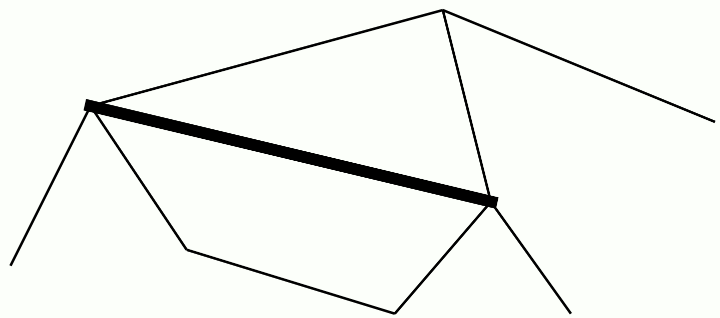


Let us see Cauchy's proof for $k = 2$.

let f be a face, b be its perimeter, p be the plane of f, and I be the point at infinity on a straight line orthogonal to p

1. pull the edges of f outwards and project all the other vertices and edges from I onto p so that all their projections are contained into f: we have now a planar graph in p with border b / one face is lost: k' = k - 1 /

2. triangulate the graph / for each new edge one new face arises: b and k' are unchanged /

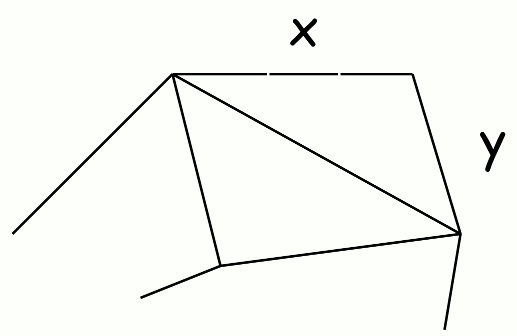


3. while (E > 3)

{ if (there is a face sharing two edges x, y with b)

remove x, y updating b ;

/ two edges, one vertex, and one face are lost: k'' = k' /



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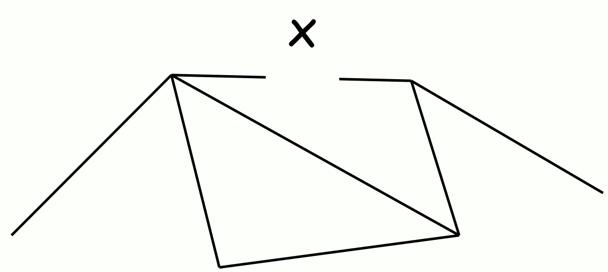


The output is one triangle for which

$$V = 3, \; E = 3, \; F = 1, \; \text{i.e.} \quad K = 1$$

Therefore we have $k=2$

for the original polyhedron.

Let us now apply Euler's formula $V - E + F = 2$

to prove that the regular polyhedra $\{ p,q \}$ are exactly five

recall that:

p is the number of edges per face,

q is the number of edges per vertex

The faces are equal regular polygons;

Each vertex is determined by the intersection of at least three faces.

The sum of the face angles at each vertex must be less than $360°$.

Hence, as observed by Plato:

$3 \leq p \leq 5$ (only triangles, squares, and pentagons may be there)

$3 \leq q \leq 5$ (three to five triangles, three squares, three pentagons)

Each face has p edges,

and each edge is shared by two faces:

$$Fp/2 = E \quad \Rightarrow \quad F = 2E/p \quad (1)$$

q edges concur in each vertex,

and each edge contains two vertices:

$$Vq/2 = E \quad \Rightarrow \quad V = 2E/q \quad (2)$$

Euler's formula is then rewritten as:

$$2E/q - E + 2E/p = 2 \quad \Rightarrow \quad E = 2pq/(2p + 2q - pq) \quad (3)$$

where the last expression (i.e. the denominator) must be >0.

It can be immediately verified that $2p + 2q - pq > 0$ only for:

$$p = 3,\ q = 3, 4, 5 \qquad p = 4,\ q = 3 \qquad p = 5,\ q = 3$$

- $p = 3, q = 3 \Rightarrow E = 6$ from (3), $V = 4$ from (1), $F = 4$ from (2) and we have the tetrahedron {3,3}
- $p = 3, q = 4 \Rightarrow E = 12, V = 6, F = 8$ the octahedron {3,4}
- $p = 3, q = 5 \Rightarrow E = 30, V = 12, F = 20$ the icosahedron {3,5}
- $p = 4, q = 3 \Rightarrow E = 12, V = 8, F = 6$ the cube {4,3}
- $p = 5, q = 3 \Rightarrow E = 30, V = 20, F = 12$ the dodecahedron {5,3}

3. GOING TO INFINITY

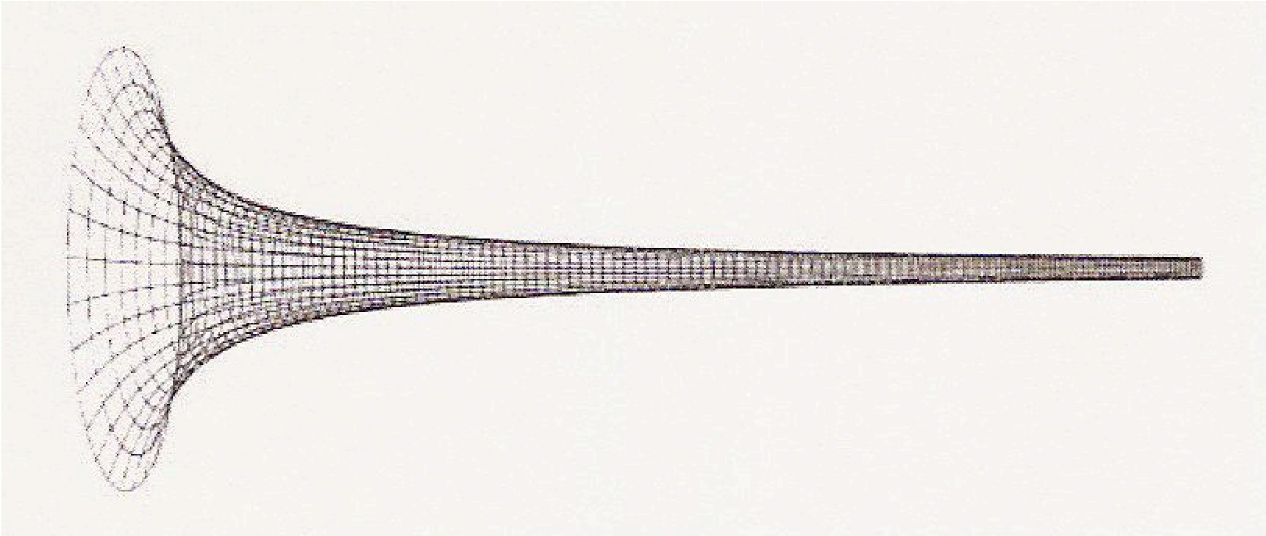
Galileo - *Discorsi intorno a due nuove scienze*
on the "paradox" of squares

1	2	3	4	5	6	. . . .
1	4	9	16	25	36	. . . .

"Infinity is out of our comprehension:
we simply cannot apply our usual reasoning"

A paradox at infinity:

Archangel Gabriel's horn



Evangelista Torricelli built this surface in 1641 by rotating a section of hyperbola around the x axis, with $x \geq 1$

using Cavalieri's techniques, he proved that the corresponding solid has volume π

The radius: $y = 1/x$

$$V = \int_1^\infty \pi(1/x^2)dx = \pi(\lim_{x\to\infty}(-1/x)+1) = \pi$$

The fact that an infinite body had finite volume was considered a paradox. But in addition:

Torricelli proved that the surface of the horn is infinite ! !

$$S = \int_1^\infty 2\pi(1/x)dx = 2\pi(\lim_{x\to\infty}(\ln x)+0) = \infty$$

The paradox caused a strong mathematical and philosophical debate in the XVII century

Cavalieri apparently doubted that his method of computing volumes could contain a bug

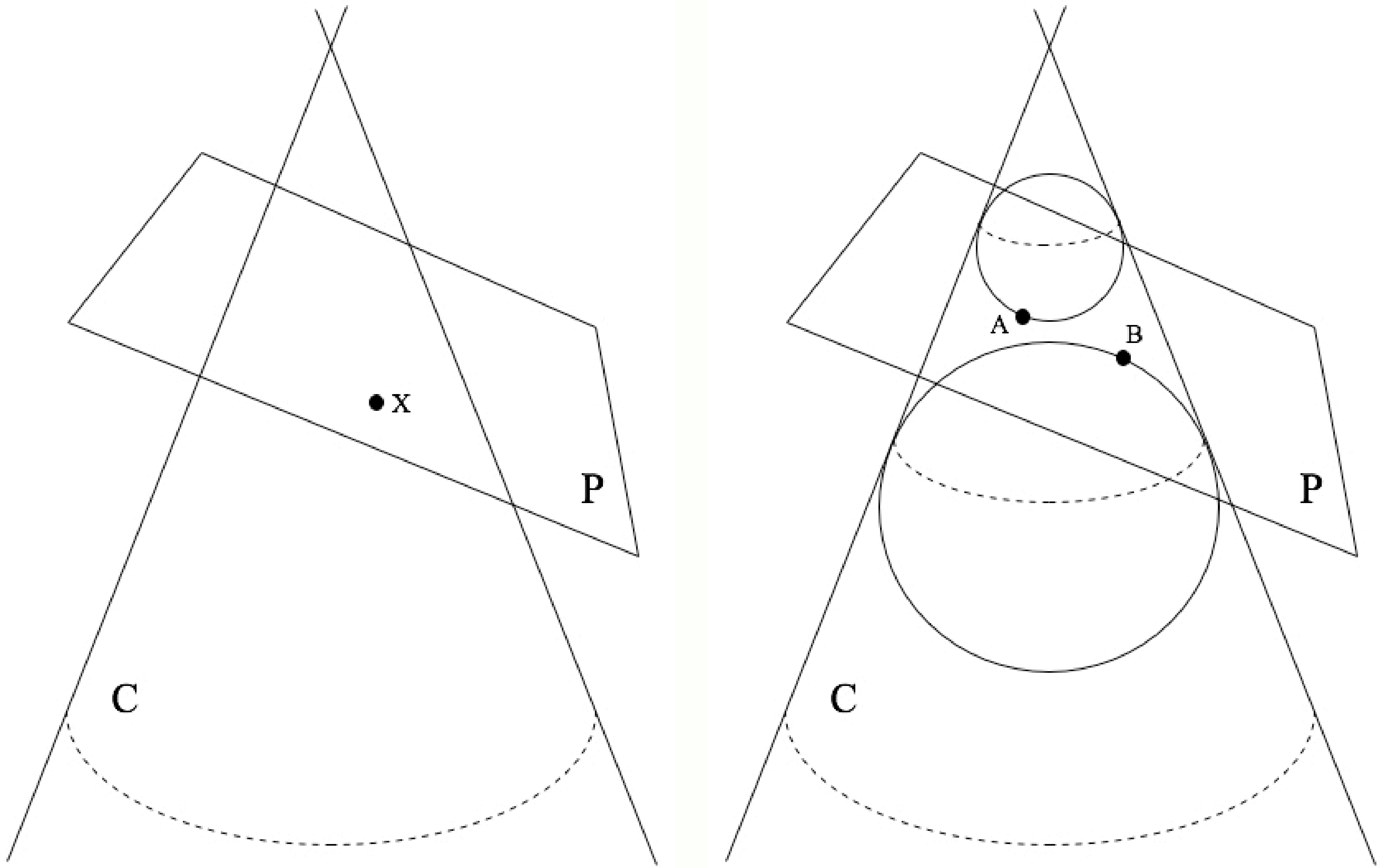
Thomas Hobbes is reported to have said:

"To understand this for sense it is not required that a man should be a geometrician or a logician, but that he should be mad"

4. INTERSECTING A RIGHT CIRCULAR CONE WITH A PLANE

Apollonius of Perga
Two centuries B.C.

The construction of Dandelin (1822) based on the metric definition of the ellipse



X
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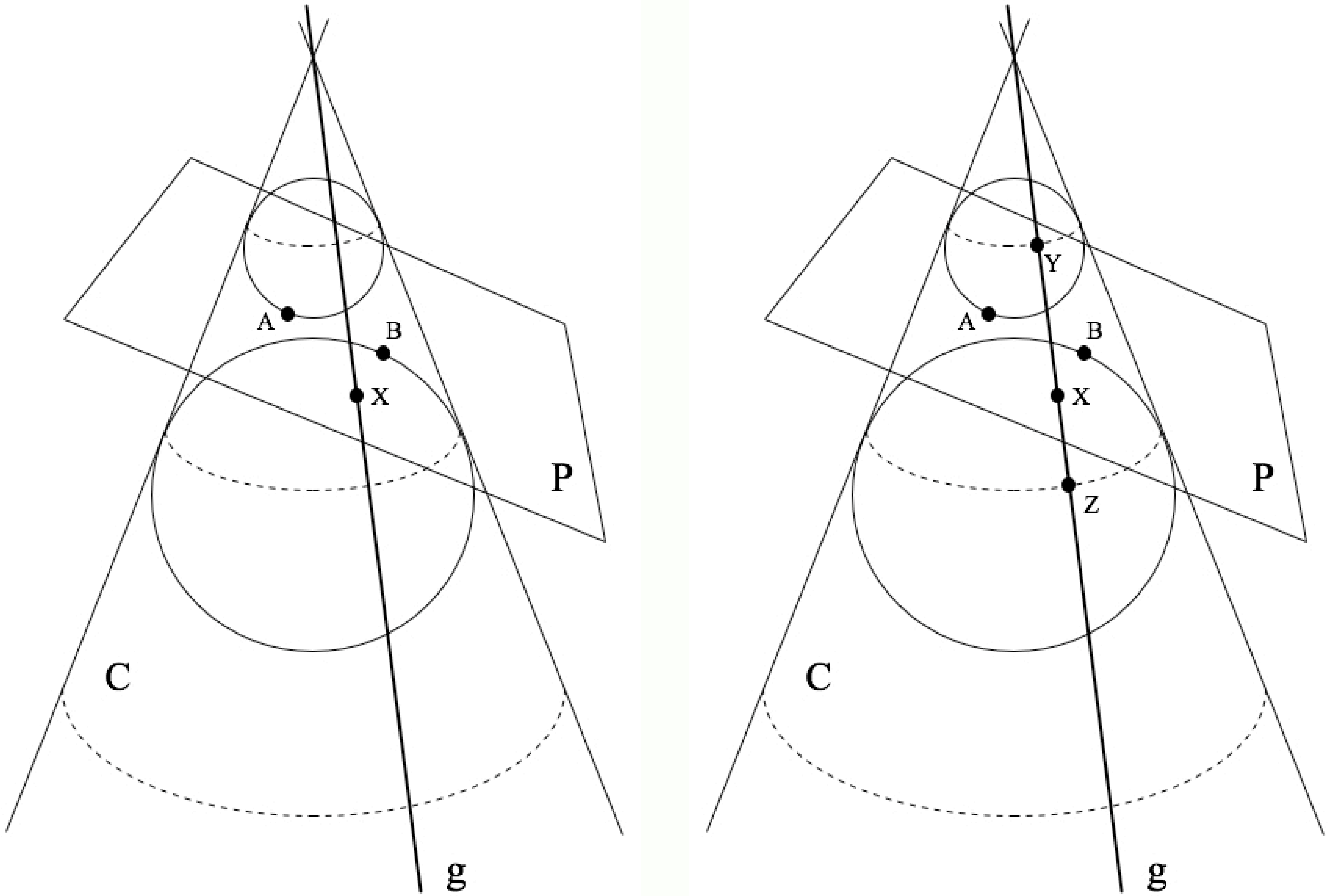
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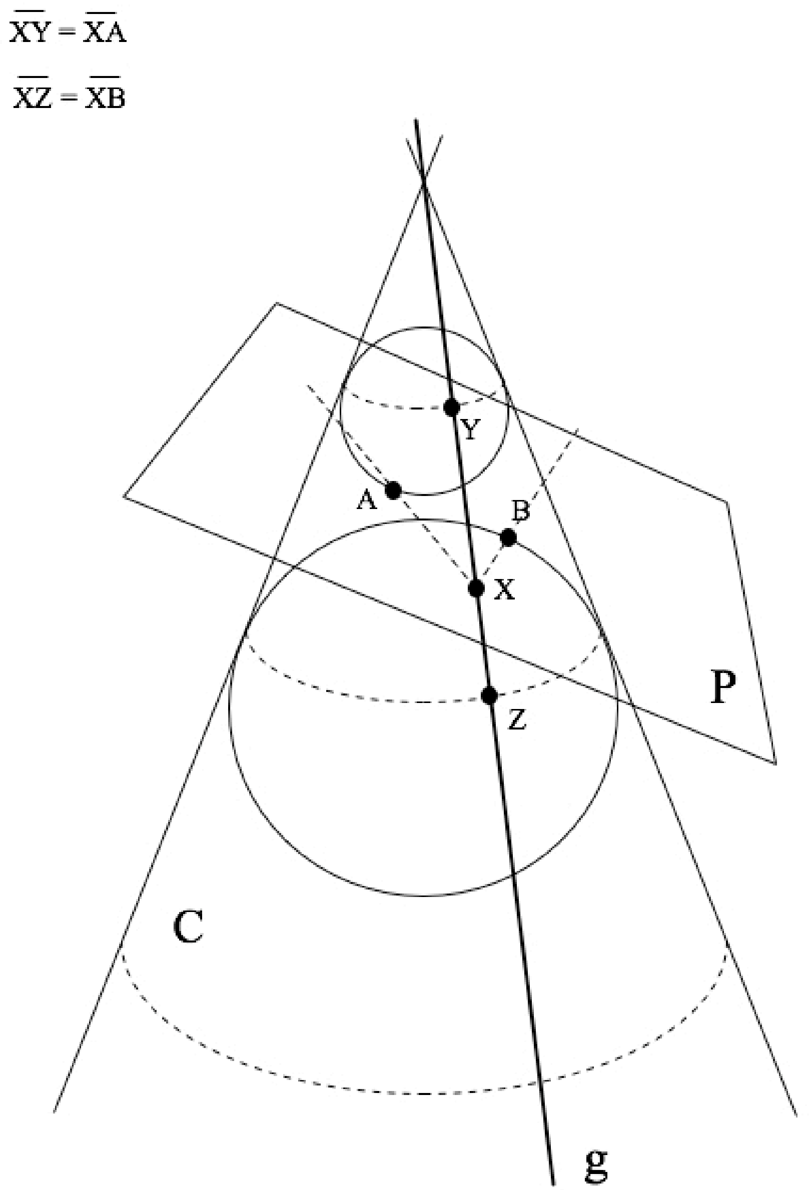
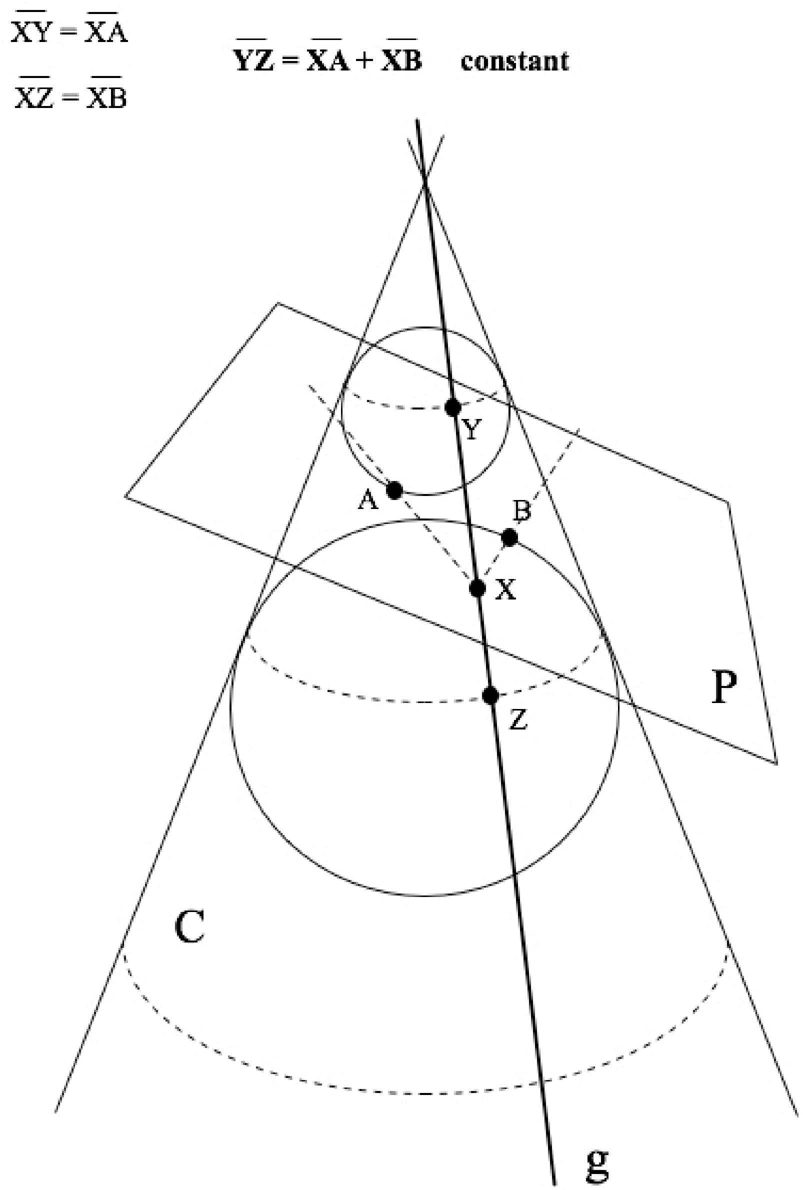

$\overline{XY} = \overline{XA}$
$\overline{XZ} = \overline{XB}$
Y
A
B
X
Z
P
C
g


$\overline{XY} = \overline{XA}$
$\overline{XZ} = \overline{XB}$
$\mathbf{\overline{YZ} = \overline{XA} + \overline{XB}}$ **constant**
Y
A
B
X
Z
P
C
g

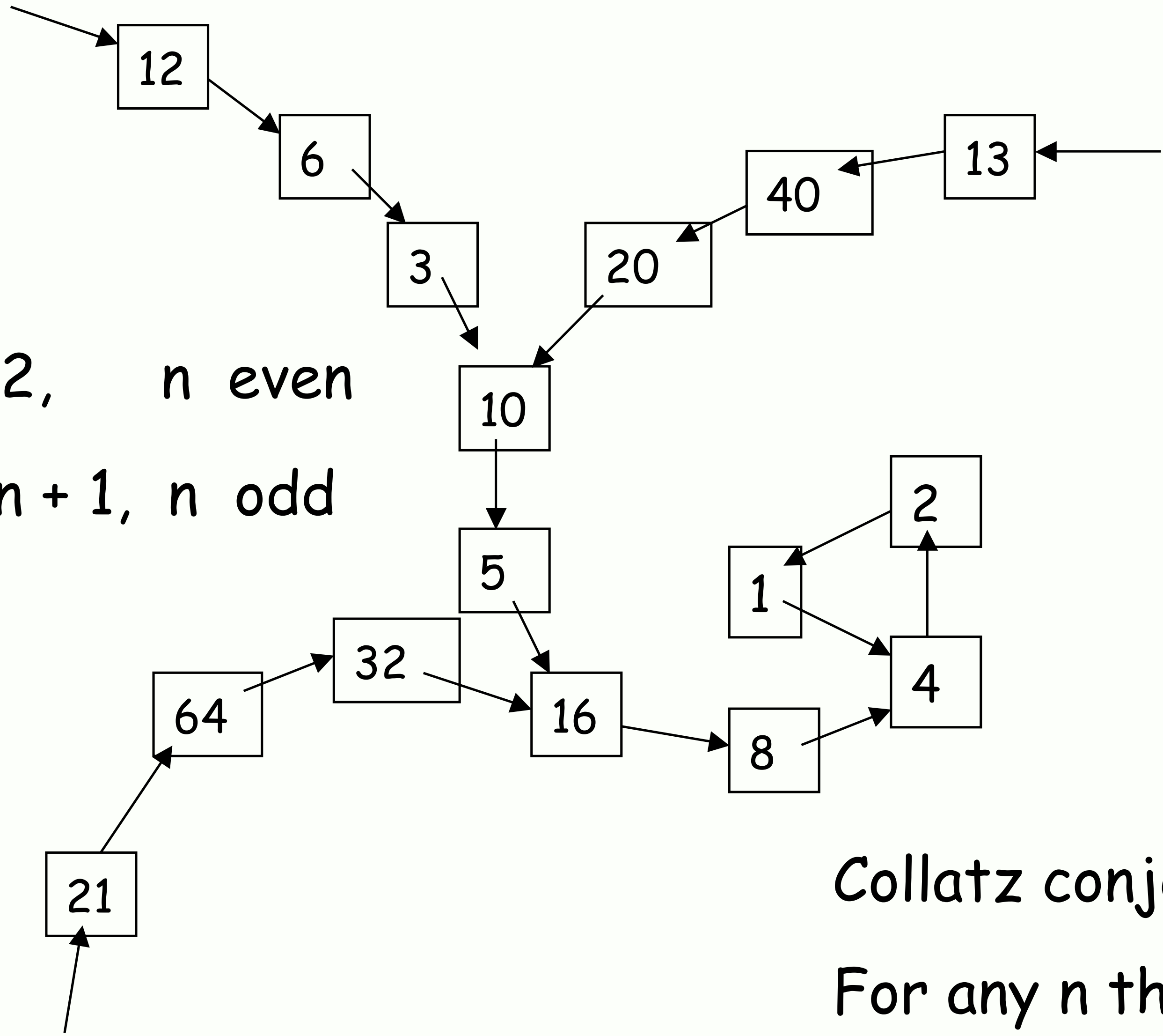
5. THREE AMAZING SEQUENCES

Collatz: A conjecture, 1937

later named after Ulam, Kakutani, Thwaites, Hasse, or called the Syracuse problem

Goodstein: The number $G_n(m)$, 1944

Kirby and Paris: The Hydra, 1982



$n' = n/2, \quad n$ even

$n' = 3n + 1, \quad n$ odd

Collatz conjecture:

For any n the series converges to 1

The sequence for n = 27 takes 111 steps, climbing to 9232 before descending to 1

27, 82, 41, 124, 62, 31, 94, 47, 142, 71, 214, 107, 322, 161, 484, 242, 121, 364, 182, 91, 274, 137, 412, 206, 103, 310, 155, 466, 233, 700, 350, 175, 526, 263, 790, 395, 1186, 593, 1780, 890, 445, 1336, 668, 334, 167, 502, 251, 754, 377, 1132, 566, 283, 850, 425, 1276, 638, 319, 958, 479, 1438, 719, 2158, 1079, 3238, 1619, 4858, 2429, 7288, 3644, 1822, 911, 2734, 1367, 4102, 2051, 6154, 3077, 9232, 4616, 2308, 1154, 577, 1732, 866, 433, 1300, 650, 325, 976, 488, 244, 122, 61, 184, 92, 46, 23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1

The Goodstein numbers

Let $m \geq 0, n > 1$ be integers

The n-representation of m, shortly $\langle n\text{-} m \rangle$, is:

1. write m as the sum of powers (SOP) of n
2. write each exponent as the SOP of n
3. repeat with exponents of exponents until all digits involved are $\leq n$

For $m = 266$ $n = 2$:

1. $266 = 2^8 + 2^3 + 2^1$
2. $266 = (2^2)^3 + 2^{2+1} + 2^1$
3. $266 = (2^2)^{2+1} + 2^{2+1} + 2^1$ this is $\langle 2\text{-}266 \rangle$

Define the Goodstein number $G_n(m)$, $n \geq 2$, as follows:

$G_n(m) = 0$ for $m = 0$;

$G_2(m) = \langle 2\text{-}m \rangle$;

$G_n(m)$, $n > 2$, is obtained by $G_{n-1}(m)$ by replacing every $n-1$ with n and subtracting 1, so that $G_n(m)$ is in n-representation

For example:

$$G_2(266) = \langle 2\text{-}266 \rangle = (2^2)^{2+1} + 2^{2+1} + 2^1$$

$$G_3(266) = (3^3)^{3+1} + 3^{3+1} + 3^1 - 1$$

$$= (3^3)^{3+1} + 3^{3+1} + 2$$

The Goodstein sequence m_k for m starting at n = 2:

$$m_0 = G_2(m), \quad m_1 = G_3(m_0), \quad m_2 = G_4(m_1), \quad \ldots\ldots$$

$$266_0 = (2^2)^{2+1} + 2^{2+1} + 2^1$$

$$266_1 = (3^3)^{3+1} + 3^{3+1} + 2 \sim 10^{38}$$

$$266_2 = (4^4)^{4+1} + 4^{4+1} + 1 \sim 10^{616}$$

$$266_3 = (5^5)^{5+1} + 5^{5+1} \sim 10^{10,000}$$

$$266_4 = (6^6)^{6+1} + 6^{6+1} - 1$$

$$= (6^6)^{6+1} + 5\cdot 6^6 + 5\cdot 6^5 + \ldots\ldots + 5\cdot 6 + 5 \sim ??$$

The sequence m_k may start at any $n \geq 2$:

$$m_0 = G_n(m), \quad m_1 = G_{n+1}(m_0), \quad m_2 = G_{n+2}(m_1), \quad . \; . \; . \; . \; . \; .$$

Theorem [Goodstein 1944].

For any m, n there is a value $k \geq 0$ such that $m_k = 0$.

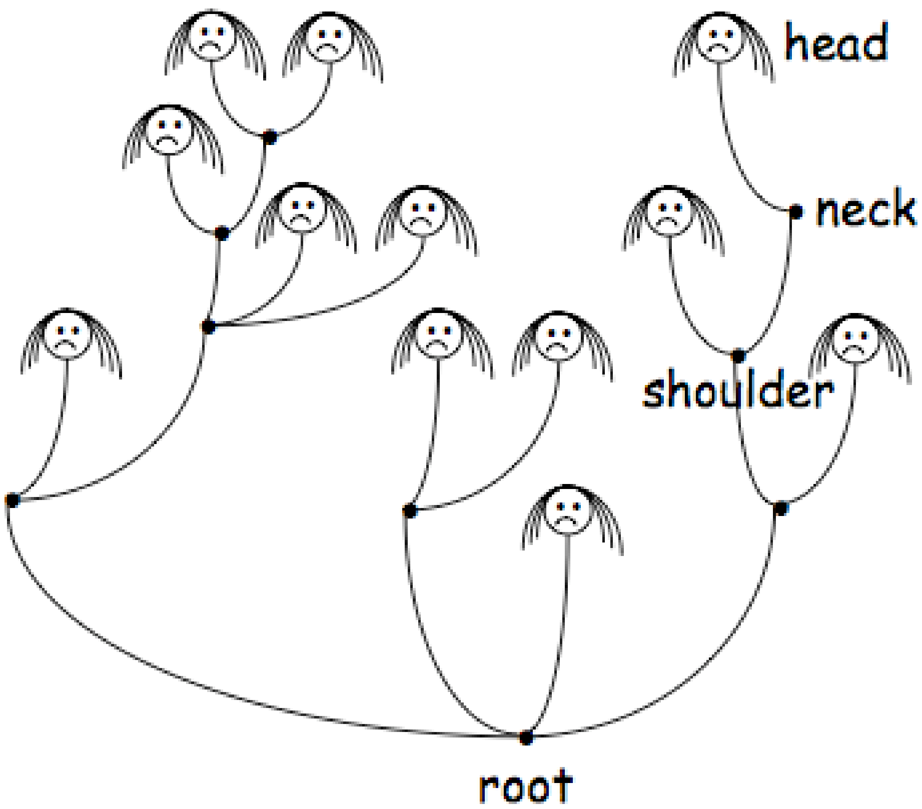
(The proof is based on transfinite induction)

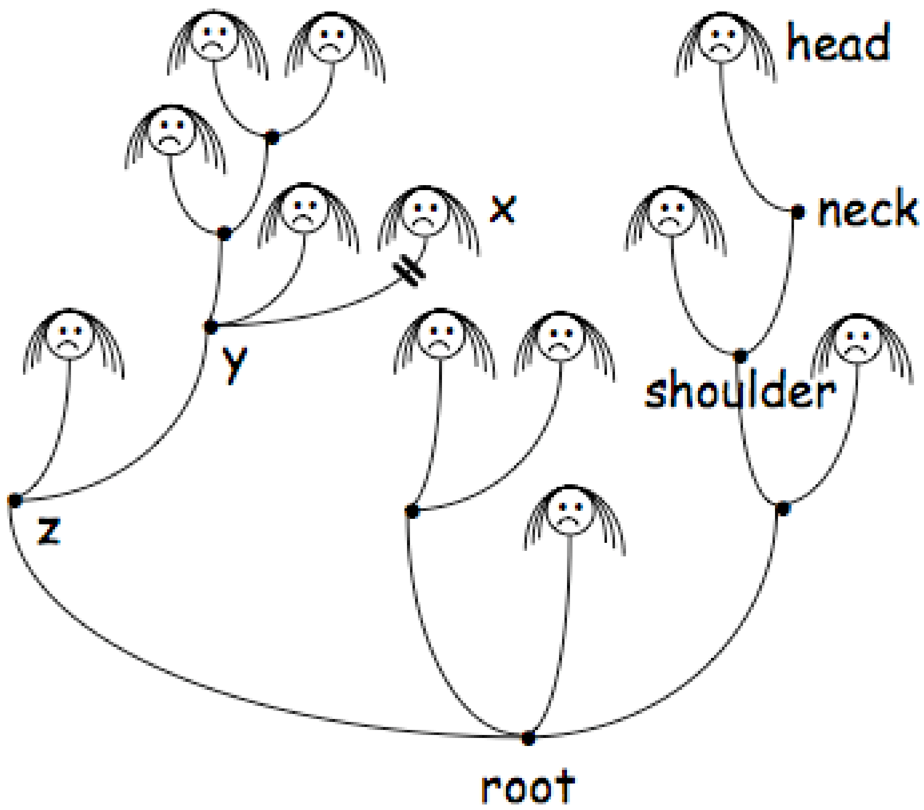
That is, the Goodstein sequence for any m, starting at any n, eventually hits 0 . . . . but very slowly

E.g., for $m = 4$ the sequence 4_k staring at $n = 2$ reaches 0 for $k = 3 \times 2^{402,653,211} - 3$

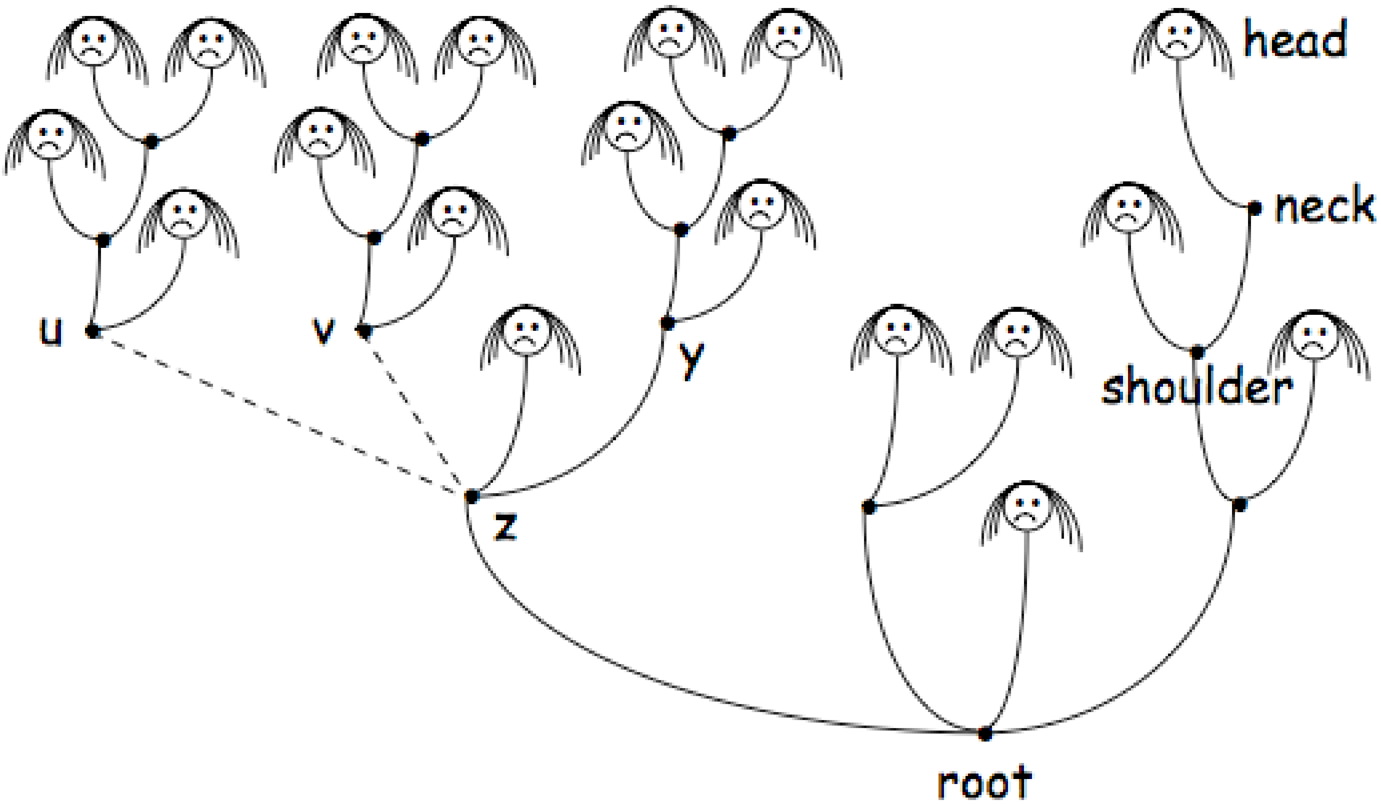
The battle between Hercules and the Hydra

A Hydra is a finite rooted tree





if Hercules chops off one head x
at each step n



n copies of the neck and heads sprout off the Hydra's shoulder (here we show step 2)

Hercules *wins* if after a finite sequence of head chopping, nothing is left of the Hydra but its root.

A *strategy* is a function which determines for Hercules which head to chop off at each step of a battle.

With ingenuity one or more winning strategies can be found. They may require different (and obviously huge) numbers of steps. More surprisingly:

Theorem [Kirby and Paris, 1982].

Every strategy for Hercules is a winning strategy.

(Again the proof is done with transfinite induction).

Even the most inexperienced Hercules cannot help winning!

Sturgeon's Law (1958)

90% of everything is crud

I hope we stayed in the 10% complementary set